A STRONGLY QUASI-HEREDITARY STRUCTURE ON AUSLANDER–DLAB–RINGEL ALGEBRAS

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Abstract. Auslander–Dlab–Ringel (ADR) algebras of semilocal modules have quasi-hereditary structures. In this note, we show that such algebras are always left-strongly quasi-hereditary. As an application, we give a better upper bound for global dimension of ADR algebras of semilocal modules. Moreover, we describe characterizations of original ADR algebras to be strongly quasi-hereditary.

1. Preliminaries

Throughout this note, A is an artin algebra and J is the Jacobson radical of A. We denote by $\mathsf{mod}A$ the category of finitely generated right A-modules and by $\mathsf{proj}A$ the category of finitely generated projective right A-modules. For $M \in \mathsf{mod}A$, we write $\mathsf{add}M$ for the category of all direct summands of finite direct sums of copies of M.

We fix a complete set of representatives of isomorphism classes of simple A-modules $\{S(i) \mid i \in I\}$. For $i \in I$, we denote by $P(i)$ the projective cover of $S(i)$ and $E(i)$ the injective hull of $S(i)$. Let $\leq$ be a partial order on I. For each $i \in I$, we denote by $\nabla(i)$ the maximal submodule of $E(i)$ whose composition factors have the form $S(j)$ for some $j \leq i$. The module $\nabla(i)$ is called the costandard module corresponding to i. Let $\nabla := \{\nabla(i) \mid i \in I\}$ be the set of standard modules. We denote by $\mathcal{F}(\nabla)$ the full subcategory of $\mathsf{mod}A$ whose objects are the modules which have a ∇-filtration.

In this section, we quickly review a relationship between strongly quasi-hereditary algebras and rejective chains. For more detail, we refer to [6, 9].

We start this section with recalling the definition of left-strongly quasi-hereditary algebras.

Definition 1 ([7, §4])**.** Let A be an artin algebra and $\leq$ a partial order on I.

(1) A pair $(A, \leq)$ (or simply A) is called a *left-strongly quasi-hereditary* algebra if there exists a short exact sequence

$$0 \to \nabla(i) \to E(i) \to E(i)/\nabla(i) \to 0$$

for any $i \in I$ with the following properties:

(a) $E(i)/\nabla(i) \in \mathcal{F}(\nabla)$ for any $i \in I$;

(b) if $(E(i)/\nabla(i) : \nabla(j)) \neq 0$, then we have $i < j$;

(c) $E(i)/\nabla(i)$ is an injective A-module, or equivalently, $\nabla(i)$ has injective dimension at most one.

The detailed version of this article has been published in [10].

(2) We say that a pair $(A, \leq)$ (or simply A) is a *right-strongly quasi-hereditary* algebra if $(A^{\mathrm{op}}, \leq)$ is left-strongly quasi-hereditary.

(3) We say that a pair $(A, \leq)$ (or simply A) is a *strongly quasi-hereditary* algebra if $(A, \leq)$ is left-strongly quasi-hereditary and right-strongly quasi-hereditary.

By definition, strongly quasi-hereditary algebras are left-strongly (resp. right-strongly) quasi-hereditary algebras. Since a pair $(A, \leq)$ satisfying the conditions (a) and (b) is a quasi-hereditary algebra, left-strongly (resp. right-strongly) quasi-hereditary algebras are quasi-hereditary.

Left-strongly (resp. right-strongly) quasi-hereditary algebras are characterized by total left (resp. right) rejective chains, which are chains of certain left (resp. right) rejective subcategories. We recall the notion of left (resp. right) rejective subcategories.

Let $\mathcal{C}$ be an additive category, and put $\mathcal{C}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)$. In the following, *we assume that any subcategory is full and closed under isomorphisms, direct sums and direct summands.*

Definition 2 ([5, 2.1(1)])**.** Let $\mathcal{C}$ be an additive category. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called

(1) a *left* (*resp. right*) *rejective subcategory* of $\mathcal{C}$ if, for any $X \in \mathcal{C}$, there exists an epic left (resp. monic right) $\mathcal{C}'$-approximation $f^X \in \mathcal{C}(X, Y)$ (resp. $f_X \in \mathcal{C}(Y, X)$) of X,

(2) a *rejective subcategory* of $\mathcal{C}$ if $\mathcal{C}'$ is a left and right rejective subcategory of $\mathcal{C}$.

To define a total left (resp. right) rejective chain, we need the notion of cosemisimple subcategories. Let $\mathcal{J}_{\mathcal{C}}$ be the Jacobson radical of $\mathcal{C}$. For a subcategory $\mathcal{C}'$ of $\mathcal{C}$, we denote by $[\mathcal{C}']$ the ideal of $\mathcal{C}$ consisting of morphisms which factor through some object of $\mathcal{C}'$, and by $\mathcal{C}/[\mathcal{C}']$ the factor category (*i.e.*, $ob(\mathcal{C}/[\mathcal{C}']) := ob(\mathcal{C})$ and $(\mathcal{C}/[\mathcal{C}'])(X, Y) := \mathcal{C}(X, Y)/[\mathcal{C}'](X, Y)$ for any $X, Y \in \mathcal{C}$). Recall that an additive category $\mathcal{C}$ is called a *Krull–Schmidt* category if any object of $\mathcal{C}$ is isomorphic to a finite direct sum of objects whose endomorphism rings are local.

Definition 3. Let $\mathcal{C}$ be a Krull–Schmidt category. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a *cosemisimple* subcategory in $\mathcal{C}$ if $\mathcal{J}_{\mathcal{C}/[\mathcal{C}']} = 0$ holds.

We give a characterization of cosemisimple left (resp. right) rejective subcategories. In the following, we denote by $\mathsf{ind}\mathcal{C}$ the set of isoclasses of indecomposable objects in $\mathcal{C}$.

Proposition 4 ([6, 1.5.1])**.** *Let $\mathcal{C}$ be a Krull–Schmidt category and let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. Then $\mathcal{C}'$ is a cosemisimple left* (*resp. right*) *rejective subcategory of $\mathcal{C}$ if and only if, for any $X \in \mathsf{ind}\mathcal{C} \setminus \mathsf{ind}\mathcal{C}'$, there exists a morphism $\varphi : X \to Y$* (*resp. $\varphi : Y \to X$*) *such that $Y \in \mathcal{C}'$ and $\mathcal{C}(Y, -) \xrightarrow{-\circ\varphi} \mathcal{J}_{\mathcal{C}}(X, -)$* (*resp. $\mathcal{C}(-, Y) \xrightarrow{\varphi\circ-} \mathcal{J}_{\mathcal{C}}(-, X)$*) *is an isomorphism on $\mathcal{C}$.*

Now, we introduce the following key notion in this note.

Definition 5 ([5, 2.1(2)])**.** Let $\mathcal{C}$ be a Krull–Schmidt category. A chain

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$$

of subcategories of $\mathcal{C}$ is called

(1) a *rejective chain* if $\mathcal{C}_i$ is a cosemisimple rejective subcategory of $\mathcal{C}_{i-1}$ for $1 \leq i \leq n$,

(2) a *total left* (resp. *right*) *rejective chain* if the following conditions hold for $1 \leq i \leq n$:
   (a) $\mathcal{C}_i$ is a left (resp. right) rejective subcategory of $\mathcal{C}$;
   (b) $\mathcal{C}_i$ is a cosemisimple subcategory of $\mathcal{C}_{i-1}$.

The following proposition gives a connection between left-strongly (resp. right-strongly) quasi-hereditary algebras and total left (resp. right) rejective chains.

Proposition 6 ([9, Theorem 3.22]). *Let A be an artin algebra. Let M be a right A-module and $B := \mathrm{End}_A(M)$. Then the following conditions are equivalent.*
 (i) *B is a left-strongly* (*resp. right-strongly*) *quasi-hereditary algebra.*
 (ii) $\mathsf{proj}B$ *has a total left* (*resp. right*) *rejective chain.*
 (iii) $\mathsf{add}M$ *has a total left* (*resp. right*) *rejective chain.*

In particular, B is strongly quasi-hereditary if and only if $\mathsf{add}M$ *has a rejective chain.*

We end this section with recalling a special total left (resp. right) rejective chain, which plays an important role in this note.

Definition 7 ([6, Definition 2.2]). Let A be an artin algebra and $\mathcal{C}$ a subcategory of $\mathsf{mod}A$. A chain
$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$$
of subcategories of $\mathcal{C}$ is called an *A-total left* (resp. *right*) *rejective chain of length n* if the following conditions hold for $1 \leq i \leq n$:
 (a) for any $X \in \mathcal{C}_{i-1}$, there exists an epic (resp. monic) in $\mathsf{mod}A$ left (resp. right) $\mathcal{C}_i$-approximation of X;
 (b) $\mathcal{C}_i$ is a cosemisimple subcategory of $\mathcal{C}_{i-1}$.

All A-total left (resp. right) rejective chains of $\mathcal{C}$ are total left (resp. right) rejective chains.

We can give an upper bound for global dimension by using A-total left (resp. right) rejective chains.

Proposition 8 ([6, Theorem 2.2.2]). *Let A be an artin algebra and M a right A-module. If* $\mathsf{add}M$ *has an A-total left* (*resp. right*) *rejective chain of length $n > 0$, then the global dimension of* $\mathrm{End}_A(M)$ *is at most n.*

2. Main result

Let A be an artin algebra with Loewy length m. In [1], Auslander studied the endomorphism algebra $B := \mathrm{End}_A(\bigoplus_{j=1}^{m} A/J^j)$ and proved that B has finite global dimension. Furthermore, Dlab and Ringel showed that B is a quasi-hereditary algebra [3]. Hence B is called an *Auslander–Dlab–Ringel algebra* (ADR) algebra. Recently, Conde gave a left-strongly quasi-hereditary structure on ADR algebras [2].

In this section, we study ADR algebras of semilocal modules introduced by Lin and Xi [3]. Recall that a module M is called a semilocal module if M is a direct sum of modules which have a simple top. Since any artin algebra is a semilocal module, the ADR algebras of semilocal modules are a generalization of the original ADR algebras. In [3], they proved that ADR algebras of semilocal modules are quasi-hereditary. We refine this result as follows.

Theorem 9 ([10, Theorem 2.2])**.** *The ADR algebra of any semilocal module is left-strongly quasi-hereditary.*

Throughout this section, suppose that M is a semilocal module with Loewy length $\ell\ell(M) = m$. We denote by $\widetilde{M}$ the basic module of $\oplus_{i=1}^{m} M/MJ^i$ and call $\mathrm{End}_A(\widetilde{M})$ the ADR algebra of M. Note that $\mathrm{End}_A(\widetilde{A})$ is an ADR algebra in the sense of [2].

In the rest of this section, we give a sketch of proof of Theorem 9. In the following, we use the following notations. Let F be the set of pairwise non-isomorphic indecomposable direct summands of $\widetilde{M}$ and F_i the subset of F consisting of all modules with Loewy length $m-i$. We denote by $\mathsf{F}_{i,1}$ the subset of F_i consisting of all modules X which do not have a surjective map in $\mathcal{J}_{\mathsf{mod}A}(X,N)$ for all modules N in F_i. For any integer $j>1$, we inductively define the subsets $\mathsf{F}_{i,j}$ of F_i as follows: $\mathsf{F}_{i,j}$ consists of all modules $X \in \mathsf{F}_i \setminus \bigcup_{1\le k\le j-1} \mathsf{F}_{i,k}$ which do not have a surjective map in $\mathcal{J}_{\mathsf{mod}A}(X,N)$ for all modules $N \in \mathsf{F}_i \setminus \bigcup_{1\le k\le j-1} \mathsf{F}_{i,k}$. We set $n_i := \min\{j \mid \mathsf{F}_i = \bigcup_{1\le k\le j} \mathsf{F}_{i,k}\}$ and $n_M := \sum_{i=0}^{m-1} n_i$. For $0 \le i \le m-1$ and $1 \le j \le n_i$, we set

$$\mathsf{F}_{>(i,j)} := \mathsf{F} \setminus ((\cup_{-1\le k\le i-1}\mathsf{F}_k) \cup (\cup_{1\le l\le j}\mathsf{F}_{i,l})),$$
$$\mathcal{C}_{i,j} := \mathsf{add} \bigoplus_{N\in\mathsf{F}_{>(i,j)}} N,$$

where $\mathsf{F}_{-1} := \emptyset$. To prove Theorem 9, we need the following proposition.

Proposition 10. *Let A be an artin algebra and M a semilocal A-module. Then $\mathsf{add}\widetilde{M}$ has the following A-total left rejective chain with length n_M.*

$$\mathsf{add}\widetilde{M} =: \mathcal{C}_{0,0} \supset \mathcal{C}_{0,1} \supset \cdots \supset \mathcal{C}_{0,n_0} \supset \mathcal{C}_{1,1} \supset \cdots \supset \mathcal{C}_{m-1,n_{m-1}} = 0.$$

Now, we are ready to show Theorem 9.

Proof of Theorem 9. By Proposition 6, it is enough to show that $\mathsf{add}\widetilde{M}$ has a total left rejective chain. Hence the assertion follows from Proposition 10. □

3. Applications

In this section, we give two applications of results of Section 2.

Firstly, we give an upper bound for a global dimension of ADR algebras.

Corollary 11. *Let A be an artin algebra and M a semilocal A-module. Then the global dimension of $\mathrm{End}_A(\widetilde{M})$ is at most n_M.*

Proof. By Proposition 10, $\mathsf{add}\widetilde{M}$ has an A-total left rejective chain with length n_M. Hence the assertion follows from Proposition 8. □

Secondly, we study a connection between ADR algebras and strongly quasi-hereditary algebras. By Theorem 9, an ADR algebra is a left-strongly quasi-hereditary algebra but not necessarily strongly quasi-hereditary. We give characterizations of original ADR algebras to be strongly quasi-hereditary.

We keep the notation of the previous section. Throughout this section, A is an artin algebra with Loewy length m and $B := \mathrm{End}_A(\widetilde{A})$ the ADR algebra of A. Then $n_j = 1$

holds for any $0 \leq j \leq m-1$. Hence we obtain the following A-total left rejective chain by Proposition 10.

$$\mathsf{add}\widetilde{A} \supset \mathcal{C}_{0,1} \supset \mathcal{C}_{1,1} \supset \cdots \supset \mathcal{C}_{m-1,1} = 0. \tag{3.1}$$

Note that if $m = 1$, then B is semisimple. Hence we always assume $m \geq 2$ in the rest of this section.

Theorem 12 ([10, Theorem 3.1])**.** *Let A be an artin algebra with Loewy length $m \geq 2$ and J the Jacobson radical of A. Let $B := \mathrm{End}_A(\bigoplus_{j=1}^{m} A/J^j)$ be the ADR algebra of A. Then the following statements are equivalent.*

(i) *B is a strongly quasi-hereditary algebra.*
(ii) *The chain* (3.1) *is a rejective chain of* $\mathsf{add}\widetilde{A}$.
(iii) $\mathrm{gldim}\, B = 2$.
(iv) $J \in \mathsf{add}(\bigoplus_{j=1}^{m} A/J^j)$.

In the rest of this section, we give a sketch of proof of Theorem 12. To prove Theorem 12, we need the following lemma.

Lemma 13. *Let A be an artin algebra. If $P(i)J \in \mathsf{add}\widetilde{A}$ for any $i \in I$, then $P(i)J/P(i)J^j \in \mathsf{add}\widetilde{A}$ for $1 \leq j \leq m$.*

Now, we are ready to prove Theorem 12.

Proof of Theorem 12. (ii) $\Rightarrow$ (i): The assertion follows from Proposition 6.

(i) $\Rightarrow$ (iii): It follows from [7, Proposition A.2] that the global dimension of B is at most two. In this case, we can check that $\mathrm{gldim}\, B = 2$.

(iii) $\Leftrightarrow$ (iv): This follows from [8, Proposition 2].

(iv) $\Rightarrow$ (ii): First, we show that $\mathcal{C}_{0,1}$ is a cosemisimple rejective subcategory of $\mathsf{add}\widetilde{A}$. By Proposition 10, it is enough to show that $\mathcal{C}_{0,1}$ is a right rejective subcategory of $\mathsf{add}\widetilde{A}$. For any $X \in \mathsf{ind}(\mathsf{add}\widetilde{A}) \setminus \mathsf{ind}(\mathcal{C}_{0,1})$, there exists an inclusion map $\varphi : XJ \hookrightarrow X$ with $XJ \in \mathcal{C}_{0,1}$ by the condition (iv). Since X is a projective A-module such that its Loewy length coincides with the Loewy length of A, the map φ induces an isomorphism

$$\mathrm{Hom}_A(\widetilde{A}, XJ) \xrightarrow{\varphi \circ -} \mathcal{J}_{\mathsf{mod}A}(\widetilde{A}, X).$$

It follows from Proposition 4 that $\mathcal{C}_{0,1}$ is a cosemisimple right rejective subcategory of $\mathsf{add}\widetilde{A}$. Hence we obtain that $\mathcal{C}_{0,1}$ is a cosemisimple rejective subcategory of $\mathsf{add}\widetilde{A}$.

Next, we prove that $\mathsf{add}\widetilde{A}$ has a rejective chain

$$\mathsf{add}\widetilde{A} \supset \mathcal{C}_{0,1} \supset \mathcal{C}_{1,1} \supset \cdots \supset \mathcal{C}_{m-1,1} = 0$$

by induction on m. If $m = 2$, then the assertion holds. Assume that $m \geq 3$. Let $X \in \mathsf{ind}(\mathcal{C}_{0,1}) \setminus \mathsf{ind}(\mathcal{C}_{1,1})$. Then $X = P(i)/P(i)J^{m-1}$ for some $i \in I$ and we have

$$(P(i)/P(i)J^{m-1})J(A/J^{m-1}) \cong P(i)J/P(i)J^{m-1}.$$

Since $P(i)J \in \mathsf{add}\widetilde{A}$, we obtain $P(i)J/P(i)J^{m-1} \in \mathcal{C}_{0,1}$ by Lemma 13. By induction hypothesis, $\mathcal{C}_{0,1}$ has the following rejective chain.

$$\mathcal{C}_{0,1} \supset \mathcal{C}_{1,1} \supset \cdots \supset \mathcal{C}_{m-1,1} = 0.$$

Composing it with $\mathsf{add}\widetilde{A} \supset \mathcal{C}_{0,1}$, we obtain a rejective chain of $\mathsf{add}\widetilde{A}$. □

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